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A SIMPLIFIED REISSNER THEORY FOR PLATE BENDING

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Abstract—A theory for plate bending is presented which includes the effects of transverse shear and transverse direct stress. It is based on the assumptions of Reissner's theory, but, by accepting a specified order of accuracy it is shown to be possible to formulate the theory for both homogeneous and sandwich plates in terms of transverse displacement as the single unknown variable. The application of finite difference and localized Ritz methods to the theory is briefly discussed.

NOTATION

D $\left(=\dfrac{Eh^3}{12(1-\nu^2)}\right)$ flexural rigidity of plate
D_x, D_y flexural rigidities of orthotropic sandwich plate in pure bending
D_1 $=\nu_{yx}D_x=\nu_{xy}D_y$
D_{xy} torsional rigidity of sandwich plate in pure twist
E Young's modulus
h plate thickness
L length of side of plate
M_x, M_y bending moments per unit length
M_{xy} twisting moment per unit length
q uniformly distributed load per unit area
Q_x, Q_y shear forces per unit length
S_x, S_y shear stiffnesses of sandwich plate
U strain energy
w transverse displacement
x, y, z orthogonal co-ordinates
Δ^2 $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)^2$
Δ^3 $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)^3$
ν Poisson's ratio
ν_{xy}, ν_{yx} Poisson's ratios for orthotropic sandwich plate
$\sigma_x, \sigma_y, \sigma_z$ normal components of stress
$\tau_{xy}, \tau_{xz}, \tau_{yz}$ shear components of stress
ϕ_x, ϕ_y average rotations
ψ Reissner stress function.

1. INTRODUCTION

The principal limitations of classical plate theory are well known, namely that no account is taken of the deformation due to transverse shear and transverse direct stress, and that only two conditions can be satisfied at each boundary whereas three exist.

There have been several attempts to improve classical theory within the confines of a two-dimensional plate theory, enabling useful results to be obtained without carrying out a fully three-dimensional analysis. The theory of Reissner[1–3] occupies a unique position amongst the literature on this subject. This theory is formulated in terms of two variables, transverse displacement, w, and a stress function, ψ, for which the governing equations are of fourth order and second order respectively. The system of equations is therefore sixth order, thus requiring three conditions to be satisfied at each boundary, and includes the effects of transverse shear and transverse direct stress.

Interest in the Reissner plate theory has continued and solutions in series form have been obtained for certain rectangular plate problems—simply supported[4], simply supported on two opposite edges with the other edges free[4], simply supported on two opposite edges with a variety of conditions on the remaining edges[5], and plates supported by an elastic foundation[6].

Series solutions are inevitably restricted to the treatment of simple geometric shapes and certain types of boundary, and hence the range of problems for which such solutions may be obtained is very limited. Further, they involve considerable mathematical complexity because of the two variable formulation.

A finite element solution based directly on Reissner's theory has been formulated[7], but with the transverse direct stress terms omitted. Transverse displacement of the rectangular element is defined by the usual twelve term polynomial, but with five degrees of freedom at each node, two of which account for shear deformation alone. By this means the need for the stress function is avoided, although explicit expressions for the stress resultants in terms of transverse displacement are not achieved.

So far a Reissner type theory has not been explicitly formulated in terms of the single variable, w, although this would appear attractive from both a computational and physical viewpoint. The purpose of this paper, therefore, is to establish the extent to which it is possible to derive a plate bending theory based on the Reissner assumptions, including shear deformation and transverse direct stress, but with the governing equation and stress resultants expressed as explicit functions of one variable, and to examine the use of numerical methods for its solution.

2. THEORETICAL DEVELOPMENT

It will be shown that a theory including the effects of both transverse shear stress and transverse direct stress can be derived in terms of transverse displacement as the single variable to a specified order of accuracy. An accuracy of order $0(h^2)$ will be regarded as adequate, with an error term of order $0(h^4)$.

Such a degree of approximation may be seen to be acceptable by considering the initial assumptions of Reissner's theory. The distributions of stress assumed through the depth of the plate are linear for bending stresses σ_x, σ_y and shear stress τ_{xy}, quadratic for transverse shear stresses τ_{xz}, τ_{yz}, and cubic for transverse direct stress, σ_z. These are known to be approximations at least in the case of the bending stresses, since when shear deformation is considered the section does not remain plane but warps. The two-dimensional formulation is preserved by working in terms of three displacements averaged through the depth of the plate—transverse displacement, w, and rotations ϕ_x and ϕ_y, the averaging being carried out from work-energy considerations.

Reissner derived the following relationships for shear and moment stress resultants for homogeneous, isotropic plates

$$Q_x - \frac{h^2}{10}\Delta Q_x = -D\left(\frac{\partial^3 w}{\partial x^2} + \frac{\partial^3 w}{\partial x \partial y^2}\right) - \frac{h^2}{10(1-\nu)}\frac{\partial q}{\partial x} \tag{1}$$

$$Q_y - \frac{h^2}{10}\Delta Q_y = -D\left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y}\right) - \frac{h^2}{10(1-\nu)}\frac{\partial q}{\partial y} \tag{2}$$

$$M_x = D\left(\frac{\partial \phi_x}{\partial x} + \nu\frac{\partial \phi_y}{\partial y}\right) + \frac{6\nu(1+\nu)}{5Eh}q \tag{3}$$

$$M_y = D\left(\frac{\partial \phi_y}{\partial y} + \nu\frac{\partial \phi_x}{\partial x}\right) + \frac{6\nu(1+\nu)}{5Eh}q \tag{4}$$

$$M_{xy} = -D\frac{(1-\nu)}{2}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \tag{5}$$

in which

$$\phi_x = -\frac{\partial w}{\partial x} + \frac{12(1+\nu)}{5Eh}Q_x \tag{6}$$

$$\phi_y = -\frac{\partial w}{\partial y} + \frac{12(1+\nu)}{5Eh}Q_y. \tag{7}$$

From the usual equilibrium relationship

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}=-q \tag{8}$$

it follows that

$$\frac{\partial q}{\partial x}=-\frac{\partial^2 Q_x}{\partial x^2}-\frac{\partial^2 Q_y}{\partial x\partial y}. \tag{9}$$

Hence eqn (1) can be written as

$$Q_x=-D\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right)+\frac{h^2}{10}\left(\frac{\partial^2 Q_x}{\partial x^2}+\frac{\partial^2 Q_x}{\partial y^2}\right)+\frac{h^2}{10(1-\nu)}\left(\frac{\partial^2 Q_x}{\partial x^2}+\frac{\partial^2 Q_y}{\partial x\partial y}\right) \tag{10}$$

and an expression for Q_y can be found in a similar manner. Differentiating these gives

$$\frac{\partial^2 Q_x}{\partial x^2}=-D\left(\frac{\partial^5 w}{\partial x^5}+\frac{\partial^5 w}{\partial x^3\partial y^2}\right)+\text{terms in } h^2$$

$$\frac{\partial^2 Q_x}{\partial y^2}=-D\left(\frac{\partial^5 w}{\partial x^3\partial y^2}+\frac{\partial^5 w}{\partial x\partial y^4}\right)+\text{terms in } h^2$$

$$\frac{\partial^2 Q_y}{\partial x\partial y}=-D\left(\frac{\partial^5 w}{\partial x\partial y^4}+\frac{\partial^5 w}{\partial x^3\partial y^2}\right)+\text{terms in } h^2.$$

Substituting these expressions in eqn (10) yields

$$Q_x=-D\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right)-\frac{h^2(2-\nu)}{10(1-\nu)}D\left(\frac{\partial^5 w}{\partial x^5}+2\frac{\partial^5 w}{\partial x^3\partial y^2}+\frac{\partial^5 w}{\partial x\partial y^4}\right)$$
$$+(\text{terms in } h^4 \text{ and higher powers of } h). \tag{11}$$

Similarly it can be shown that

$$Q_y=-D\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2\partial y}\right)-\frac{h^2(2-\nu)}{10(1-\nu)}D\left(\frac{\partial^5 w}{\partial y^5}+2\frac{\partial^5 w}{\partial x^2\partial y^3}+\frac{\partial^5 w}{\partial x^4\partial y}\right)$$
$$+(\text{terms in } h^4 \text{ and higher powers of } h). \tag{12}$$

Accepting that $0(h^2)$ is an adequate order of accuracy, and ignoring terms in h^4 and higher powers of h from hereon, substitution of (11) and (12) into (8) gives as the governing equation for plate deflexion

$$\Delta^2 w+\frac{h^2(2-\nu)}{10(1-\nu)}\Delta^3 w=\frac{q}{D} \tag{13}$$

Substituting eqns (11) and (12) in (6) and (7), and then the resulting expressions for ϕ_x and ϕ_y into (3)–(5) and noting (13) gives the following relationships for bending and twisting moments to order of accuracy $0(h^2)$

$$M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)-\frac{h^2 D}{10(1-\nu)}\left((2-\nu)\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\nu\frac{\partial^4 w}{\partial y^4}\right) \tag{14}$$

$$M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)-\frac{h^2 D}{10(1-\nu)}\left((2-\nu)\frac{\partial^4 w}{\partial y^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\nu\frac{\partial^4 w}{\partial x^4}\right) \tag{15}$$

$$M_{xy}=(1-\nu)D\frac{\partial^2 w}{\partial x\partial y}+\frac{h^2 D}{5}\left(\frac{\partial^4 w}{\partial x^3\partial y}+\frac{\partial^4 w}{\partial x\partial y^3}\right). \tag{16}$$

Thus the state of deformation and the shear and moment stress resultants are defined by eqns (11)–(16) solely in terms of transverse displacement, with an error term of order $0(h^4)$.

3. SOLUTION USING NUMERICAL METHODS

Solutions to this theory have been obtained using both finite differences and the localized Ritz method.

In the finite difference approach the usual central differences of accuracy (mesh length)^2 are used. Since the governing eqn (13) is sixth order, three fictitious mesh points will be associated with each boundary point and hence three boundary conditions must be satisfied.

The application of the localized Ritz method to classical plate bending theory has been described by Walker[8]. In using the method with the present theory ten degrees of freedom are required at each node, namely freedom of

$$w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}, \frac{\partial^3 w}{\partial x^3}, \frac{\partial^3 w}{\partial x^2 \partial y}, \frac{\partial^3 w}{\partial x \partial y^2}, \frac{\partial^3 w}{\partial y^3}.$$

The appropriate strain energy function is

$$U = \frac{1}{2E}\int\int\int \{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z) + 2(1+\nu)(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)\}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z. \tag{17}$$

Omitting the term in σ_z^2 and rewriting in terms of stress resultants this becomes

$$U = \frac{1}{2E}\int\int \left\{\frac{12}{h^3}(M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)M_{xy}^2) + \frac{12(1+\nu)}{5h}(Q_x^2 + Q_y^2) - \frac{12\nu q}{5h}\left(M_x + M_y\right)\right\} \mathrm{d}x\, \mathrm{d}y \tag{18}$$

which may then be expressed in terms of derivatives of w by substituting from eqns (11), (12) and (14)–(16). The total potential energy is then minimized with respect to each coefficient associated with the generalised displacements at the nodes.

4. AN ILLUSTRATIVE EXAMPLE

As an example of the results obtained using the above theory the simply supported square plate carrying a uniformly distributed load is considered here. This case is chosen since a series solution of Reissner's theory is also available[4], from which numerical results have been calculated for purposes of comparison.

The central deflexion of the plate for a range of depth/span ratios is shown in Fig. 1 as a ratio of the value given by classical bending theory. The modification in deflexion predicted by the theory is then clearly seen in relation to that part of it which is due to bending alone.

Both solutions follow the series solution fairly accurately, but tend to overestimate the deflexion for thick plates. At $h/L = 0.3$ the differences are 18 and 12% for the finite difference and localized Ritz solutions respectively. Simpler versions of this theory have been developed for beams, and analytical solutions have shown that both theories then yield identical results. It is therefore believed that the differences in the results for plates arise from the numerical analysis rather than from the approximations introduced in the simplified theory. The fact that the difference between the two sets of numerical results is of the same order as the error reinforces this view.

There is a clear conceptual advantage in working in terms of transverse displacement as the single variable, as there is no physical basis for interpretation of the Reissner stress function. Further, certain computational difficulties have been found to be associated with the stress function when numerical methods are applied to Reissner's theory directly.

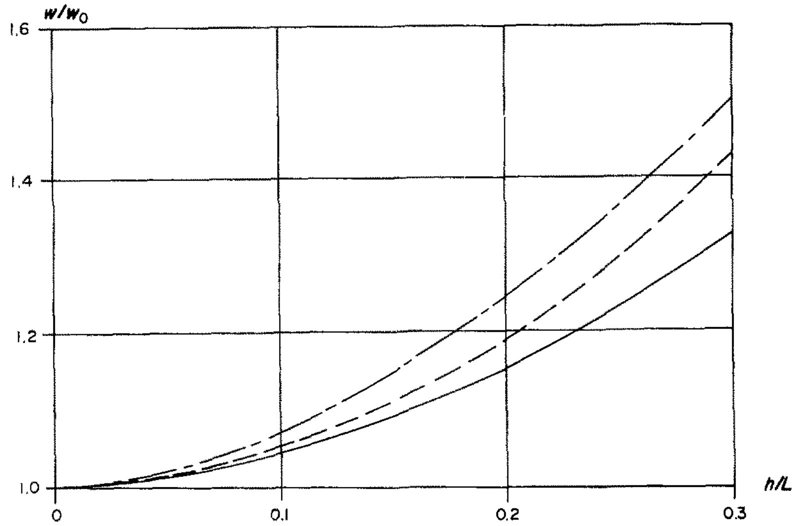

Fig. 1. Central deflexion ratio. Simply supported square plate: uniform load ($\nu = 0$) ($w_0 = 0.00406\ qL^4/D$). (——), Reissner—series solution. (—·—), Simplified theory—finite difference solution. (-----), Simplified theory—localized Ritz solution.

5. APPLICATION TO SANDWICH PLATES

Shear deformation is of particular interest in sandwich plates and the simplified theory can readily be applied to this class of problem. It is conventional for such structures to assume that the effects of transverse direct stress are negligible. Omitting these terms, eqns (3)–(7) become, in orthotropic form,

$$M_x = D_x \frac{\partial \phi_x}{\partial x} + D_1 \frac{\partial \phi_y}{\partial y} \tag{19}$$

$$M_y = D_y \frac{\partial \phi_y}{\partial y} + D_1 \frac{\partial \phi_x}{\partial x} \tag{20}$$

$$M_{xy} = -D_{xy}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \tag{21}$$

$$\phi_x = -\frac{\partial w}{\partial x} + \frac{1}{S_x} Q_x \tag{22}$$

$$\phi_y = -\frac{\partial w}{\partial y} + \frac{1}{S_y} Q_y. \tag{23}$$

The appropriate equations for stress resultants in accordance with the assumptions of the simplified Reissner theory are obtained by the following process:

(a) Substitute eqns (22) and (23) in eqns (19)–(21) to give M_x, M_y, M_{xy} as functions of w, Q_x, Q_y.

(b) Obtain expressions Q_x, Q_y analogous to eqn (10) by substituting the equations for M_x, M_y, M_{xy} from (a) into the equilibrium relationships

$$Q_x = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y}$$

$$Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x}$$

(c) follow the procedure of repeated differentiation of the equations for Q_x, Q_y and back substitution as described above for isotropic homogeneous plates.

The resulting equations are

$$M_x = -Dx\frac{\partial^2 w}{\partial x^2} - D_1\frac{\partial^2 w}{\partial y^2} - \frac{D_x^2}{S_x}\frac{\partial^4 w}{\partial x^4} + (D_1 + 2D_{xy})\left(\frac{D_x}{S_x} + \frac{D_1}{S_y}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{D_1 D_y}{S_y}\frac{\partial^4 w}{\partial y^4} \tag{24}$$

$$M_y = -D_y\frac{\partial^2 w}{\partial y^2} - D_1\frac{\partial^2 w}{\partial x^2} - \frac{D_y^2}{S_y}\frac{\partial^4 w}{\partial y^4} - (D_1 + 2D_{xy})\left(\frac{D_y}{S_y} + \frac{D_1}{S_x}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{D_1 D_x}{S_x}\frac{\partial^4 w}{\partial x^4} \tag{25}$$

$$M_{xy} = 2D_{xy}\frac{\partial^2 w}{\partial x \partial y} + D_{xy}\left(\frac{D_x}{S_x} + \frac{D_1 + 2D_{xy}}{S_y}\right)\frac{\partial^4 w}{\partial x^3 \partial y} + D_{xy}\left(\frac{D_1 + 2D_{xy}}{S_x} + \frac{D_y}{S_y}\right)\frac{\partial^4 w}{\partial x \partial y^3} \tag{26}$$

$$\begin{aligned} Q_x = &- D_x\frac{\partial^3 w}{\partial x^3} - (D_1 + 2D_{xy})\frac{\partial^3 w}{\partial x \partial y^2} - \frac{D_x^2}{S_x}\frac{\partial^5 w}{\partial x^5} - \left(\frac{D_x(D_1 + 3D_{xy})}{S_x} + (D_1 + D_{xy})(D_1 + 2D_{xy})\right)\frac{\partial^5 w}{\partial x^3 \partial y^2} \\ &- (D_1 + 2D_{xy})\left(\frac{D_{xy}}{S_x} + \frac{(D_1 + D_{xy})}{S_y}\right)\frac{\partial^5 w}{\partial x \partial y^4} \end{aligned} \tag{27}$$

$$\begin{aligned} Q_y = &- D_y\frac{\partial^3 w}{\partial y^3} - (D_1 + 2D_{xy})\frac{\partial^3 w}{\partial x^2 \partial y} - \frac{D_y^2}{S_y}\frac{\partial^5 w}{\partial y^5} - \left(\frac{D_y(D_1 + 3D_{xy})}{S_y} + (D_1 + D_{xy})(D_1 + 2D_{xy})\right)\frac{\partial^5 w}{\partial x^2 \partial y^3} \\ &- (D_1 + 2D_{xy})\left(\frac{D_{xy}}{S_y} + \frac{(D_1 + D_{xy})}{S_x}\right)\frac{\partial^5 w}{\partial x^4 \partial y}. \end{aligned} \tag{28}$$

The governing equation for w is then found by substituting eqns (27) and (28) in eqn (8). This

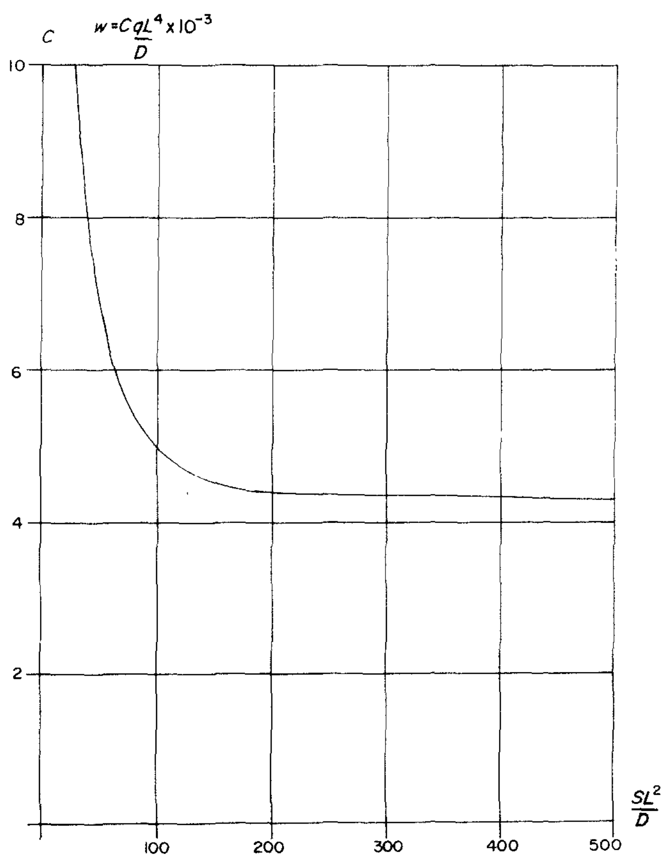

Fig. 2. Central deflexion of simply supported isotropic sandwich plate subjected to uniform loading (——).

single variable formulation is in contrast to other commonly used sandwich plate theories in terms of three variables, either w, Q_x and Q_y [9] or partial deflexions [10].

Figure 2 shows the central deflexion results for the simply supported square plate carrying uniform load appropriate to an isotropic sandwich with

$$D_x = D_y = D$$

$$S_x = S_y = S$$

$$\nu_{xy} = \nu_{yx} = 0$$

for a range of values of SL^2/D.

The central deflexion is seen to become very large for low values of shear stiffness, while tending to the classical bending value of $0.00406qL^4/D$ as the shear stiffness becomes very large.

In this case the governing equilibrium equation and the equations for stress resultants for a given value of S/D are the same as those for a homogeneous plate with the same ratio of shear stiffness to bending stiffness. In the homogeneous case $S = 5Gh/6$, giving $S/D = 5/h^2$ when $\nu = 0$. This comparison is not valid if $\nu \neq 0$ since the deflexions of the homogeneous plate are then modified by the effects of transverse direct stress, which have been neglected in the sandwich formulation.

6. CONCLUSIONS

By accepting a specified order of accuracy it has been demonstrated that the equations related to plate bending including the effects of both shear and transverse displacement as the only variable. These equations have been stated to order of accuracy $0(h^2)$, with an error term of order $0(h^4)$, and it has been shown that numerical solutions can be obtained using finite difference and localized Ritz methods.

A corresponding theory has been formulated for sandwich plates, and results obtained for one case over a practical range of ratios of shear stiffness to bending stiffness.

REFERENCES

1. E. Reissner, On the theory of bending of elastic plates. *J. Math. Phys.* **23**, 184 (1944).
2. E. Reissner, The effect of transverse shear deformation on the bending of elastic plates. *J. Appl. Mech.* **12**, A69 (1945).
3. E. Reissner, On bending of elastic plates. *Quart. Appl. Math.* **5**, 55 (1947).
4. V. L. Salerno and M. A. Goldberg, Effect of shear deformations on the bending of rectangular plates. *J. Appl. Mech.* **27**, 54 (1960).
5. R. C. Koeller and F. Essenburg, Shear deformation in rectangular plates. *Proc. 4th U.S. Nat. Cong. Appl. Mech.* **1**, 555 (1962).
6. D. Frederick, Thick rectangular plates on an elastic foundation. *Trans. A.S.C.E.* **122**, 1067 (1957).
7. C. W. Pryor, R. M. Barker and D. Frederick, Finite element bending analysis of Reissner plates. *J. Eng. Mech. Div. A.S.C.E.* **96**, **EM6**, 967 (1970).
8. A. C. Walker, Rayleigh–Ritz method for plate flexure. *J. Eng. Mech. Div. A.S.C.E.* **93**, **EM6**, 139 (1967).
9. C. Libove and S. B. Batdorf, A general small-deflection theory for sandwich plates. *N.A.C.A. Report No.* 899, Washington (1948).
10. F. J. Plantema, *Sandwich Construction.* Wiley, New York (1966).